Numerical Analysis of Higher Order Discontinuous Galerkin Finite Element methods

Ralf Hartmann

Institute of Aerodynamic and Flow Technology
DLR (German Aerospace Center)

13. Oct. 2008

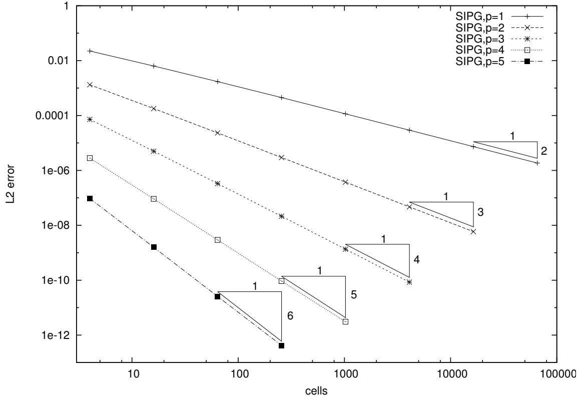
Outline

Higher order discretization methods

A discretization method is of order n if the discretization error behaves like $\mathcal{O}(h^n)$. This means:

- Reducing the mesh size from h to $h/2$ (one global mesh refinement step), the discretization error is reduced by a factor of 2^n.

Example:



DG discretization of Poisson's equation:

The $L^2(\Omega)$-error of the DG(p), $p = 1, \ldots, 5$, discretization behaves like $\mathcal{O}(h^{p+1})$

Discontinuous Galerkin Discretization

Basic properties:

- finite element method with discontinuous trial and test functions
- uses numerical flux functions
- has a local and global conservation property
- DG of 1st order is comparable to a basic finite volume method
- higher order simply by increasing the polynomial degree p
- higher order on unstructured and locally refined meshes
- different polynomial degree in different parts of the domain
- allows error estimation, hp-refinement

Outline

Important topics in the numerical analysis of finite element methods

... which **will be** covered in this lecture:

- Consistency and Galerkin orthogonality
- Coercivity and stability
- Convergence and order of convergence
- Order of convergence in specific target quantities $J(\cdot)$
- Adjoint consistency
- *A priori* error estimation

Important topics in the numerical analysis of finite element methods

... which **will be** covered in this lecture:

- Consistency and Galerkin orthogonality
- Coercivity and stability
- Convergence and order of convergence
- Order of convergence in specific target quantities $J(\cdot)$
- Adjoint consistency
- *A priori* error estimation

... which **will not be** covered in this lecture

- *A posteriori* error estimation
- Derivation of indicators for local mesh refinement (h-refinement)
- Derivation of indicators for hp-refinement

The problem and its discretization

Primal problem: Consider a linear PDE of the form

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, where L denotes a linear differential operator on Ω, and B denotes a linear differential (boundary) operator on the boundary Γ.

The problem and its discretization

Primal problem: Consider a linear PDE of the form

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, where L denotes a linear differential operator on Ω, and B denotes a linear differential (boundary) operator on the boundary Γ.

Consider the finite element **discretization**: find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

V_h is a discrete function space and $B_h : V \times V \to \mathbb{R}$ is a bilinear form.
Here V is a function space such that $V_h \subset V$ and $u \in V$, where u is the exact, i.e. analytical, solution to the primal problem.

Consistency and Galerkin orthogonality

The discretization: find $u_h \in V_h \subset V$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

is **consistent** if the exact solution $u \in V$ to the primal problem satisfies

$$B_h(u, v) = F_h(v) \quad \forall v \in V.$$

This answers the question: Do we solve the right equations?

Subtracting both equations for $v_h \in V_h \subset V$ we obtain
the **Galerkin orthogonality**:

$$B_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Coercivity & Stability

Coercivity of B_h: Is there a constant $\gamma > 0$, such that

$$B_h(v_h, v_h) \geq \gamma |||v_h|||^2 \quad \forall v_h \in V_h,$$

where $|||v|||$ is a norm (or seminorm) on V.
Continuity of F_h: Is there a constant $C_F > 0$ such that

$$F_h(v_h) \leq C_F |||v_h||| \quad \forall v_h \in V_h.$$

Then, for the solution $u_h \in V_h$ to the discrete problem

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

we obtain

$$\gamma |||u_h|||^2 \leq B_h(u_h, u_h) = F_h(u_h) \leq C_F |||u_h|||,$$

and thus **stability**: $|||u_h||| \leq \frac{C_F}{\gamma}$

If $|||\cdot|||$ is a norm (and not only a semi-norm) on V then the discretization is **stable**.

Convergence and order of convergence

- Does the discrete solution u_h converge to the exact solution u?

Convergence and order of convergence

- Does the discrete solution u_h converge to the exact solution u?
- What is the order of convergence, i.e., given a solution u with $\|u\|_{**} < \infty$, what is (the maximum) r such that

$$\|u - u_h\|_* \leq ch^r \|u\|_{**}.$$

  - Here, $\|\cdot\|_*$ is an appropriate (global) norm to measure the error in, e.g. $\|\cdot\|_* = \|\cdot\|_{L^2}$,
  - and $\|\cdot\|_{**}$ is a norm on (possibly a subset of) V.

Convergence in specific target quantities $J(\cdot)$

The target quantity $J(u)$ may represent a physically relevant quantity

- weighted mean value of the solution
- weighted boundary integral of the solution or its normal derivative
- aerodynamic force coefficients: drag, lift and moment coefficients

Given a solution u with $\|u\|_{**} < \infty$, what is (the maximum) s such that

$$|J(u) - J(u_h)| \leq ch^s \|u\|_{**}.$$

Duality based error estimates and adjoint consistency

Error estimates in the L^2-norm or in target quantities $J()$ require the use of duality arguments:

- Define an appropriate adjoint problem connected to the primal problem and the L^2-norm or the target quantity.
- Some analysis reveals that the discretization is of optimal order only if the discretization is adjoint consistent.

In addition to **consistency** require **adjoint consistency** for optimality

Adjoint consistency

Consider the primal problem

$$Lu = f \quad \text{in } \Omega, \qquad\qquad Bu = g \quad \text{on } \Gamma,$$

and the target functional

$$J(u) = \int_\Omega j_\Omega \, u \, \mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma \, u \, \mathrm{d}s,$$

with $j_\Omega \in L^2(\Omega)$ and $j_\Gamma \in L^2(\Gamma)$, and the corresponding adjoint problem

$$L^* z = j_\Omega \quad \text{in } \Omega, \qquad\qquad B^* z = j_\Gamma \quad \text{on } \Gamma.$$

The discretization: find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

is **adjoint consistent** if the solution z to the adjoint problem satisfies:

$$B_h(w, z) = J(w) \quad \forall w \in V.$$

A priori and a posteriori error estimates

A priori error estimates: e.g.

$$\begin{aligned}\|u-u_h\|_* &\le ch^r\|u\|_{**},\\ |J(u)-J(u_h)| &\le ch^s\|u\|_{**}\end{aligned}$$

A priori and a posteriori error estimates

A priori error estimates: e.g.

$$\|u - u_h\|_* \leq ch^r \|u\|_{**},$$
$$|J(u) - J(u_h)| \leq ch^s \|u\|_{**}$$

A posteriori error estimates: e.g.

$$|J(u) - J(u_h)| \leq E(u_h),$$
$$|J(u) - J(u_h)| \approx E(u_h, z_h)$$

Outline

A priori error estimates for higher order continuous and discontinuous Galerkin discretizations of model equations

Numerical analysis including

- the discretizations
- consistency, Galerkin orthogonality
- conservation properties
- coercivity, stability
- *a priori* error estimates

of higher order continuous and discontinuous finite elements methods including

- the standard (continuous) FE method for Poisson's equation
- the standard Galerkin method for the linear advection equation
- the streamline diffusion FE method for the linear advection equation

A priori error estimates for higher order continuous and discontinuous Galerkin discretizations of model equations

Numerical analysis including

- the discretizations
- consistency, Galerkin orthogonality
- conservation properties
- coercivity, stability
- *a priori* error estimates

of higher order continuous and discontinuous finite elements methods including

- the standard (continuous) FE method for Poisson's equation
- the standard Galerkin method for the linear advection equation
- the streamline diffusion FE method for the linear advection equation
- DG discretizations of the linear advection equation
- DG discretizations of Poisson's equation

Outline

1. Outline

2. Introduction
   - Higher Order Discontinuous Galerkin Finite Element methods
   - Numerical analysis of finite element methods...
   - Outline: This lecture
   - Outline: Lectures 2 and 3

3. Higher order continuous FE methods
   - Continuous FEM for Poisson's equation
   - Continuous FEM for the linear advection equation

4. Higher order DG discretizations
   - DG discretizations of the linear advection equation
   - DG discretizations of Poisson's equation

Lecture 2: Duality-based error estimation for DG methods
Lecture 3: Adjoint consistency analysis for DG discretizations of compressible flows

- Framework for analyzing consistency and adjoint consistency for linear problems
- Adjoint consistency analysis for the DG discretizations of
  - the linear advection equation
  - Poisson's equation
- *A priori* error estimates for target quantities $J(\cdot)$
- Numerical results

Lecture 2: Duality-based error estimation for DG methods
Lecture 3: Adjoint consistency analysis for DG discretizations of compressible flows

- Framework for analyzing consistency and adjoint consistency for linear problems
- Adjoint consistency analysis for the DG discretizations of
  - the linear advection equation
  - Poisson's equation
- *A priori* error estimates for target quantities $J(\cdot)$
- Numerical results

- Framework for analyzing consistency and adjoint consistency for nonlinear problems
- Adjoint consistency analysis for the DG discretizations of
  - the compressible Euler equations
  - the compressible Navier-Stokes equation
- Numerical results

Outline

Poisson's equation: The homogeneous Dirichlet problem

Consider the homogeneous Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \Gamma_D.$$

Multiply by a test function $v \in H_0^1(\Omega)$, integrate over Ω, integrate by parts

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \mathbf{n} \cdot \nabla u \, v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}\mathbf{x}.$$

The boundary term vanishes due to $v \in H_0^1(\Omega)$.
The **weak formulation**: find $u \in V := H_0^1(\Omega)$ such that

$$B(u, v) = F(v) \qquad \forall v \in V,$$

where

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x}, \qquad\qquad F(v) = \int_\Omega f v \, \mathrm{d}\mathbf{x}.$$

Poisson's equation: The inhomogeneous Dirichlet problem

Consider the **in**homogeneous Dirichlet problem,

$$-\Delta u = f \quad \text{in } \Omega, \qquad\qquad u = g_D \quad \text{on } \Gamma_D,$$

with $g_D \in L^2(\Gamma)$ on $\Gamma_D = \Gamma$, and $g_D \not\equiv 0$.
Assume that there is a $u_D \in H^1(\Omega)$ with $u_D = g_D$ on Γ_D.
The **weak formulation**: find $u = u_D + u_0$ with $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} \qquad \forall v \in H_0^1(\Omega).$$

Poisson's equation: The Neumann problem

Consider the Neumann problem,

$$-\Delta u = f \quad \text{in } \Omega, \qquad\qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

with $\Gamma_N = \Gamma$ and $g_N \in L^2(\Gamma_N)$. Multiply by a test function $v \in H^1(\Omega)$, integrate over Ω, integrate by parts

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \mathbf{n} \cdot \nabla u \, v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}\mathbf{x}.$$

The **weak formulation**: find $u \in V := H^1(\Omega)$ such that

$$B(u, v) = F(v) \quad \forall v \in V,$$

with

$$B(u, v) \equiv \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x}, \qquad\qquad F(v) = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} g_N \, v \, \mathrm{d}s.$$

Lax-Milgram theorem (Existence and Uniqueness)

Let the linear form $F : V \to \mathbb{R}$ be *continuous*, i.e. there is a $C_F > 0$ such that

$$F(v) \leq C_F \|v\|_V \quad \forall v \in V.$$

Let the bilinear form $B : V \times V \to \mathbb{R}$ be *continuous*, i.e. there is a $C_B > 0$ such that

$$B(u, v) \leq C_B \|u\|_V \|v\|_V \quad \forall u, v \in V.$$

Let B be V-coercive, i.e. there is a constant $\gamma > 0$ such that

$$B(v, v) \geq \gamma \|v\|_V^2, \quad \forall v \in V.$$

Then, there is a unique solution $u \in V$ such that

$$B(u, v) = F(v) \quad \forall v \in V.$$

We say: this problem is well-posed.

Poisson's equation: The Dirichlet-Neumann problem

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

with the weak formulation

$$B(u, v) = F(v) \quad \forall v \in V,$$

where V is a function space with $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.
Define the space of continuous piecewise polynomials of degree p:

$$\begin{aligned} V_{h,p}^c = \{v_h \in C^0(\Omega) : & v_h|_\kappa \circ \sigma_\kappa \in Q_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit hypercube, and} \\ & v_h|_\kappa \circ \sigma_\kappa \in P_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit simplex}, \kappa \in \mathcal{T}_h\}, \end{aligned}$$

Replacing u and v by discrete functions $u_h, v_h \in V_h := V_{h,p}^c$ gives the **discrete problem:** Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Analysis of the continuous FEM for Poisson's equation

Consistency:

$$B(u, v) = F(v) \quad \forall v \in V,$$

Galerkin orthogonality:

$$B(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

We have continuity of $F(\cdot)$ and of $B(\cdot, \cdot)$:

$$F(v) \leq C_F \|v\|_V \quad \forall v \in V, \qquad \text{and } B(u, v) \leq C_B \|u\|_V \|v\|_V \quad \forall u, v \in V,$$

and coercivity of $B(\cdot, \cdot)$:

$$B(v, v) \geq \gamma \|v\|_V^2 \quad \forall v \in V.$$

Thus also continuity and coercivity for all $v_h \in V_h \subset V$.
Employ Lax-Milgram theorem and obtain existence and uniqueness of the solution $u_h \in V_h$ to the discrete problem: find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Best approximation property

The discrete problem: find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h.$$

By using coercivity, Galerkin orthogonality and continuity we have

$$\begin{aligned}
\gamma \|u - u_h\|^2_{H^1(\Omega)} &\leq B(u - u_h, u - u_h) \\
&= B(u - u_h, u - v_h) + B(u - u_h, v_h - u_h) \\
&= B(u - u_h, u - v_h) \\
&\leq C_B \|u - u_h\|_{H^1(\Omega)} \|u - v_h\|_{H^1(\Omega)}
\end{aligned}$$

and obtain the **best approximation property** (Céa Lemma):

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_B}{\gamma} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Best approximation property

The discrete problem: find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h.$$

By using coercivity, Galerkin orthogonality and continuity we have

$$\begin{aligned}
\gamma \|u - u_h\|^2_{H^1(\Omega)} &\leq B(u - u_h, u - u_h) \\
&= B(u - u_h, u - v_h) + B(u - u_h, v_h - u_h) \\
&= B(u - u_h, u - v_h) \\
&\leq C_B \|u - u_h\|_{H^1(\Omega)} \|u - v_h\|_{H^1(\Omega)}
\end{aligned}$$

and obtain the **best approximation property** (Céa Lemma):

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_B}{\gamma} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Choose $v_h = I_h u \in V_h$, where I_h is the interpolation operator onto V_h.

The interpolation operator

Definition (Interpolation operator $I^c_{h,p}$ onto $V^c_{h,p}$)

For $p \geq 1$ let ϕ_i, $0 \leq i < N_h$, be a nodal basis of $V^c_{h,p}$ with

$$\phi_i(\mathbf{x}^{(j)}) = \delta_{ij} \quad \forall\, 0 \leq i, j < N_h,$$

where $\mathbf{x}^{(j)}$, $0 \leq j < N_h$, are the nodal points of $V^c_{h,p}$. For $u \in H^2(\Omega)$, we define $I^c_{h,p}u \in V^c_{h,p}$ to be the interpolation of u onto $V^c_{h,p}$ given by

$$I^c_{h,p}u(\mathbf{x}) = \sum_{0 \leq i < N_h} u(\mathbf{x}_i)\phi_i(\mathbf{x}).$$

Possible nodal basis functions are the Lagrange interpolation polynomials $\phi_i(\mathbf{x}) := L_i^{(p)}(\mathbf{x})$. In the following write I_h instead of $I^c_{h,p}$ for short.

Interpolation/approximation estimates

Let $p \geq 1$ and I_h be the interpolation operator onto $V^c_{h,p}$.
Then for $u \in H^{p+1}(\Omega)$ we have

$$\|u - I_h u\|_{H^m(\Omega)} \leq Ch^{p+1-m}|u|_{H^{p+1}(\Omega)}.$$

In particular, for $m = 0$:

$$\|u - I_h u\|_{L^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)},$$

and for $m = 1$:

$$\|u - I_h u\|_{H^1(\Omega)} \leq Ch^{p}|u|_{H^{p+1}(\Omega)}.$$

Thus the interpolation error in the

- $L^2(\Omega)$-norm behaves like $\mathcal{O}(h^{p+1})$
- $H^1(\Omega)$-norm behaves like $\mathcal{O}(h^{p})$

A priori error estimate in the H^1-norm for the continuous FEM of Poisson's equation

The best approximation property (Céa Lemma) for $V_h = V^c_{h,p}$:

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_B}{\gamma} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Choose $v_h = I_h u \in V_h$, where I_h is the interpolation operator onto V_h.

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C_B}{\gamma} \|u - I_h u\|_{H^1(\Omega)},$$

$$\boxed{\|u - u_h\|_{H^1(\Omega)} \leq Ch^p |u|_{H^{p+1}(\Omega)}}$$

Thereby, the discretization error in the H^1-norm is of $\mathcal{O}(h^p)$.

Example: For $u \in H^2(\Omega)$ and $u_h \in V^c_{h,1}$ we have the standard result:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}.$$

A priori error estimate in the L^2-norm for the continuous FEM of Poisson's equation

Employ a duality argument (Aubin-Nitsche trick):
For $v \in L^2(\Omega)$ let z be the solution to following dual/**adjoint** problem:
find $z \in H_0^1(\Omega)$ such that

$$B(w, z) = \int_\Omega wv \,\mathrm{d}\mathbf{x} \quad \forall w \in H_0^1(\Omega).$$

We assume that $z \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\|z\|_{H^2} \leq C\|v\|_{L^2}$.
Now choosing $v = w = e = u - u_h \in H_0^1(\Omega)$ yields

$$\|e\|_{L^2(\Omega)}^2 = \int_\Omega e^2 \,\mathrm{d}\mathbf{x} = B(e, z) = B(e, z - z_h) \leq C\|u - u_h\|_{H^1(\Omega)}\|z - z_h\|_{H^1(\Omega)},$$

Choosing $z_h = I_h z \in V_h$ and using the interpolation estimate we obtain

$$\|e\|_{L^2(\Omega)}^2 \leq C\|u - u_h\|_{H^1(\Omega)} h\|z\|_{H^2(\Omega)} \leq Ch\|u - u_h\|_{H^1(\Omega)}\|e\|_{L^2(\Omega)}.$$

Thus we obtain $\boxed{\|u - u_h\|_{L^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}}$

Continuous FEM for Poisson's equation: Summary

The standard Galerkin FE discretization of Poisson's equation.
A priori error estimates:

- The discretization error in the H^1-norm behaves like $\mathcal{O}(h^p)$.
- The discretization error in the L^2-norm behaves like $\mathcal{O}(h^{p+1})$.

Derivation of the L^2-error estimate required a **duality argument** including the definition of an appropriate **adjoint problem**.

Outline

The linear advection equation

Consider the linear advection equation

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \Gamma_-,$$

where $f \in L^2(\Omega)$, $\mathbf{b} \in [C^1(\Omega)]^d$, $c \in L^\infty(\Omega)$ and $g \in L^2(\Gamma_-)$.

$$\Gamma_- = \{\mathbf{x} \in \Gamma, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$$

denotes the inflow part of the boundary $\Gamma = \partial\Omega$. Multiply by a test function $v \in H^{1,\mathbf{b}}(\Omega)$ and integrate over the domain Ω,

$$\int_\Omega (\nabla \cdot (\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} = \int_\Omega fv \,\mathrm{d}\mathbf{x} \quad \forall v \in H^{1,\mathbf{b}}(\Omega).$$

Function space

$$H^{1,\mathbf{b}}(\Omega) = \{u \in L^2(\Omega) : Lu = \nabla \cdot (\mathbf{b}u) + cu \in L^2(\Omega)\} \subset H^1(\Omega).$$

Variational formulation with weak boundary conditions

$$\int_\Omega \nabla\cdot(\mathbf{b}u)\, v \,\mathrm{d}\mathbf{x} + \int_\Omega cuv \,\mathrm{d}\mathbf{x} = \int_\Omega fv \,\mathrm{d}\mathbf{x} \quad \forall v \in H^{1,\mathbf{b}}(\Omega).$$

Integrate by parts and replace u by g on Γ_-:

$$-\int_\Omega (\mathbf{b}u)\cdot\nabla v \,\mathrm{d}\mathbf{x} + \int_\Omega cuv \,\mathrm{d}\mathbf{x} + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s = \int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s,$$

where $\Gamma_+ = \Gamma \setminus \Gamma_-$ is the outflow part of the boundary.
Integrating back by parts we obtain following **variational formulation**:
find $u \in H^{1,\mathbf{b}}(\Omega)$ such that

$$\int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s = \int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s,$$

for all $v \in H^{1,\mathbf{b}}(\Omega)$

The standard Galerkin FE method with weak boundary conditions

Variational formulation: Find $u \in H^{1,\mathbf{b}}(\Omega)$ such that

$$B(u,v) = F(v) \quad \forall v \in H^{1,\mathbf{b}}(\Omega),$$

where

$$B(u,v) = \int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s,$$

$$F(v) = \int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s.$$

Replace $u, v \in H^{1,\mathbf{b}}(\Omega)$ by $u_h, v_h \in V^c_{h,p} \subset H^{1,\mathbf{b}}(\Omega)$ gives the discrete problem:
Find $u_h \in V^c_{h,p}$ such that

$$B(u_h, v_h) = F(v_h) \quad \forall v \in V^c_{h,p}.$$

The standard Galerkin FE method with weak boundary conditions

Variational formulation: Find $u \in H^{1,\mathbf{b}}(\Omega)$ such that

$$B(u,v) = F(v) \quad \forall v \in H^{1,\mathbf{b}}(\Omega),$$

where

$$B(u,v) = \int_\Omega (\nabla \cdot (\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s,$$

$$F(v) = \int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s.$$

Replace $u, v \in H^{1,\mathbf{b}}(\Omega)$ by $u_h, v_h \in V^c_{h,p} \subset H^{1,\mathbf{b}}(\Omega)$ gives the discrete problem:
Find $u_h \in V^c_{h,p}$ such that

$$B(u_h, v_h) = F(v_h) \quad \forall v \in V^c_{h,p}.$$

Consistent, thereby we have Galerkin orthogonality.

Standard Galerkin FEM: Coercivity

For any $v \in H^{1,\mathbf{b}}(\Omega)$ we have

$$B(v,v) \geq \gamma_0 \|v\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}| v^2 \,\mathrm{d}s.$$

Hence, $B(,)$ is coercive with respect to

$$\gamma_0 \|v\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}| v^2 \,\mathrm{d}s$$

However, this is not a norm on $H^{1,\mathbf{b}}(\Omega)$!
In particular, there are functions $v \in L^2(\Omega)$ with $\|v\|^2_{L^2(\Omega)} + \int_\Gamma |\mathbf{b}\cdot\mathbf{n}| v^2 \,\mathrm{d}s < \infty$ but with unbounded directional derivatives, i.e. $\|\mathbf{b}\cdot\nabla v\| = \infty$, hence $v \notin H^{1,\mathbf{b}}(\Omega)$.

Thereby: The standard Galerkin method for the linear advection equation is **unstable**

Standard Galerkin FEM: A priori error estimate

For $p \geq 1$ let $u \in H^{p+1}(\Omega)$ be the solution to the linear advection equation and $u_h \in V_{h,p}^c$ the solution to

$$B(u_h, v_h) = F(v_h) \quad \forall v \in V_{h,p}^c.$$

Then

$$\boxed{\|u - u_h\|_{L^2(\Omega)} + \left(\int_\Gamma |\mathbf{b} \cdot \mathbf{n}| \, |u - u_h|^2 \, \mathrm{d}s \right)^{1/2} \leq Ch^p |u|_{H^{p+1}(\Omega)}.}$$

For $u \in H^{s+1}(\Omega)$ with $s < p$, i.e. for reduced smoothness, we have only

$$\boxed{\|u - u_h\|_{L^2(\Omega)} + \left(\int_\Gamma |\mathbf{b} \cdot \mathbf{n}| \, |u - u_h|^2 \, \mathrm{d}s \right)^{1/2} \leq Ch^s |u|_{H^{s+1}(\Omega)}.}$$

- Convergence at rate $\mathcal{O}(h^p)$ for smooth solutions, $u \in H^{p+1}(\Omega)$
- but **no convergence** for non-smooth solutions $u \in H^1(\Omega)$

Streamline diffusion FEM for the linear advection equation

Weak formulation: Find $u \in H^{1,\mathbf{b}}(\Omega)$ such that

$$B(u,v) = F(v) \quad \forall v \in H^{1,\mathbf{b}}(\Omega),$$

where

$$B(u,v) = \int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s,$$

$$F(v) = \int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s.$$

Replacing u by a discrete function $u_h \in V_h$, v by $v_h + \delta h \mathbf{b}\cdot\nabla v_h$ on Ω, and v by v_h on Γ, gives the **discrete problem**: find $u_h \in V^c_{h,p}$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h,$$

where

$$B_h(u,v) = \int_\Omega (\nabla\cdot(\mathbf{b}u) + cu)\,(v + \delta h \mathbf{b}\cdot\nabla v)\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, uv \,\mathrm{d}s,$$

$$F_h(v) = \int_\Omega f\,(v + \delta h \mathbf{b}\cdot\nabla v)\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, gv \,\mathrm{d}s.$$

Streamline diffusion FEM: Coercivity

$$B_h(u,v) = B(u,v) + \int_\Omega (\nabla\cdot\mathbf{b}u + \mathbf{b}\cdot\nabla u + cu)\,\delta h\mathbf{b}\cdot\nabla v\,\mathrm{d}\mathbf{x},$$

where $B(\cdot,\cdot)$ is as for the standard Galerkin discretization.
The additional term

$$\delta h\int_\Omega(\mathbf{b}\cdot\nabla u)(\mathbf{b}\cdot\nabla v)\,\mathrm{d}\mathbf{x}$$

represents artificial viscosity (diffusion) in the streamline direction **b**.
Coercivity:

$$B_h(v,v) \ge \gamma\left(h\|\mathbf{b}\cdot\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \int_\Gamma |\mathbf{b}\cdot\mathbf{n}|v^2\,\mathrm{d}s\right) = \|v\|^2_{H^{1,\mathbf{b}}(\Omega)},$$

with the $H^{1,\mathbf{b}}(\Omega)$-norm:

$$\|v\|_{H^{1,\mathbf{b}}(\Omega)} = \left(h\|\mathbf{b}\cdot\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \int_\Gamma |\mathbf{b}\cdot\mathbf{n}|v^2\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

Streamline diffusion FEM: Stability

Using coercivity of B_h and continuity of F_h we obtain

$$\gamma \|u_h\|^2_{H^{1,\mathbf{b}}(\Omega)} \leq B_h(u_h, u_h) = F_h(u_h) \leq C_F \|u_h\|_{H^{1,\mathbf{b}}(\Omega)}.$$

Hence, we have

$$h\|\mathbf{b}\cdot\nabla u_h\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Omega)} + \int_\Gamma |\mathbf{b}\cdot\mathbf{n}| u_h^2 \, \mathrm{d}s = \|u_h\|^2_{H^{1,\mathbf{b}}(\Omega)} \leq \left(\frac{C_F}{\gamma}\right)^2,$$

- We have control over $\|\mathbf{b}\cdot\nabla u_h\|$.
- The streamline diffusion FE method is **stable**.

Recall: for the standard Galerkin FE method we had no control over $\|\mathbf{b}\cdot\nabla u_h\|$

Streamline diffusion FEM: A priori error estimate

For $p \geq 1$ let $u \in H^{p+1}(\Omega)$ be the solution to the linear advection equation and $u_h \in V_{h,p}^c$ the solution to

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v \in V_{h,p}^c.$$

Then

$$\boxed{\|u - u_h\|_{H^{1,\mathbf{b}}(\Omega)} \leq Ch^{p+1/2}|u|_{H^{p+1}(\Omega)}.}$$

For $u \in H^{s+1}(\Omega)$ with $s < p$, i.e. for reduced smoothness, we have only

$$\boxed{\|u - u_h\|_{H^{1,\mathbf{b}}(\Omega)} \leq Ch^{s+1/2}|u|_{H^{s+1}(\Omega)}.}$$

- Convergence at rate $\mathcal{O}(h^{p+1/2})$ for smooth solutions, $u \in H^{p+1}(\Omega)$
- Convergence at rate $\mathcal{O}(h^{1/2})$ for non-smooth solutions $u \in H^1(\Omega)$

Continuous FEM for the linear advection equation: Summary

Stability and a priori error estimates:

standard Galerkin FEM	no control over $\|\mathbf{b}\cdot\nabla u_h\|$	unstable	$\mathcal{O}(h^p)$
streamline diffusion FEM	control over $\|\mathbf{b}\cdot\nabla u_h\|$	stable	$\mathcal{O}(h^{p+1/2})$

Outline

A variational formulation of the linear advection equation

Consider the linear advection equation:

$$Lu := \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \Gamma_-.$$

Instead of

$$H^{1,\mathbf{b}}(\Omega) = \{u \in L^2(\Omega) : Lu = \nabla \cdot (\mathbf{b}u) + cu \in L^2(\Omega)\}$$

consider the broken (mesh dependent) function space

$$H^{1,\mathbf{b}}(\mathcal{T}_h) = \{u \in L^2(\Omega) : Lu = \nabla \cdot (\mathbf{b}u) + cu|_\kappa \in L^2(\kappa), \kappa \in \mathcal{T}_h\}.$$

Multiply by a test function $v \in H^{1,\mathbf{b}}(\mathcal{T}_h)$, integrate over κ

$$\int_\kappa (\nabla \cdot (\mathbf{b}u) + cu)\, v \,\mathrm{d}\mathbf{x} = \int_\kappa fv \,\mathrm{d}\mathbf{x},$$

and integrate by parts

$$-\int_\kappa (\mathbf{b}u) \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_\kappa cuv \,\mathrm{d}\mathbf{x} + \int_{\partial\kappa} \mathbf{b} \cdot \mathbf{n}\, uv \,\mathrm{d}s = \int_\kappa fv \,\mathrm{d}\mathbf{x}.$$

A variational formulation of the linear advection equation

$$-\int_\kappa (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\kappa cuv\,\mathrm{d}\mathbf{x} + \int_{\partial\kappa} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s = \int_\kappa fv\,\mathrm{d}\mathbf{x}.$$

Sum over all $\kappa \in \mathcal{T}_h$ and replace u on Γ_- by g: find $u \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ s.t.

$$\begin{aligned} -\int_\Omega (\mathbf{b}u)\cdot\nabla_h v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s \\ = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h). \end{aligned}$$

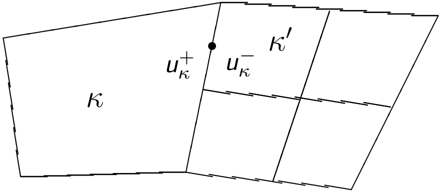
A variational formulation of the linear advection equation

$$-\int_\kappa (\mathbf{b}u)\cdot\nabla v\,\mathrm{d}\mathbf{x} + \int_\kappa cuv\,\mathrm{d}\mathbf{x} + \int_{\partial\kappa} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s = \int_\kappa fv\,\mathrm{d}\mathbf{x}.$$

Sum over all $\kappa \in \mathcal{T}_h$ and replace u on Γ_- by g: find $u \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ s.t.

$$\begin{aligned} -\int_\Omega (\mathbf{b}u)\cdot\nabla_h v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s + \int_{\Gamma_+} \mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s \\ = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h). \end{aligned}$$

Replace $\mathbf{b}\cdot\mathbf{n}\,u$ on $\partial\kappa$ by a numerical flux function $\mathcal{H}(u^+, u^-, \mathbf{n})$. $u^+ := u_\kappa^+$ and $u^- := u_\kappa^-$ are the interior and exterior traces of u on $\partial\kappa$.



A variational formulation of the linear advection equation

The variational formulation is given by: find $u \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ such that

$$B_h(u,v) = F(v) \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h),$$

with

$$\begin{aligned} B_h(u,v) = &- \int_\Omega (\mathbf{b}u) \cdot \nabla_h v \,\mathrm{d}\mathbf{x} + \int_\Omega cuv \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathcal{H}(u^+, u^-, \mathbf{n}) v \,\mathrm{d}s \\ &+ \int_{\Gamma_+} \mathbf{b} \cdot \mathbf{n}\, uv \,\mathrm{d}s, \\ F(v) = &\int_\Omega fv \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b} \cdot \mathbf{n}\, gv \,\mathrm{d}s. \end{aligned}$$

Here $\mathcal{H}(u^+, u^-, \mathbf{n})$ is a numerical flux function.

Properties of a numerical flux function

Definition:
A numerical flux function $\mathcal{H}(u^+, u^-, \mathbf{n})$ is said to be **consistent** if

$$\mathcal{H}(u, u, \mathbf{n}) = \mathbf{b} \cdot \mathbf{n}\, u.$$

Furthermore, $\mathcal{H}(u^+, u^-, \mathbf{n})$ is said to be **conservative** if

$$\mathcal{H}(u^+, u^-, \mathbf{n}) = -\mathcal{H}(u^-, u^+, -\mathbf{n}).$$

Consistency

Integration by parts on the variational formulation gives:
find $u \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ such that

$$\int_\Omega (f - \nabla_h \cdot (\mathbf{b}u) - cu)\, v \,\mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \left(\mathbf{b}\cdot\mathbf{n}\, u^+ - \mathcal{H}(u^+, u^-, \mathbf{n})\right)\, v \,\mathrm{d}s$$
$$- \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, \left(g - u^+\right) v \,\mathrm{d}s = 0 \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h).$$

For the exact solution $u \in H^{1,\mathbf{b}}(\Omega) \subset H^{1,\mathbf{b}}(\mathcal{T}_h)$ we obtain

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} (\mathbf{b}\cdot\mathbf{n}\, u - \mathcal{H}(u, u, \mathbf{n}))\, v \,\mathrm{d}s = 0 \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h).$$

Hence we have **consistency**: i.e.

$$B_h(u, v) = F(v) \quad \forall v \in H^{1,\mathbf{b}}(\mathcal{T}_h),$$

holds for the exact solution u **if and only if** $\mathcal{H}(u, u, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\, u$.

Global conservation property

Setting $c = 0$ and $v \equiv 1$ in the variational formulation we obtain

$$\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\mathcal{H}(u^+,u^-,\mathbf{n})\,\mathrm{d}s+\int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,u\,\mathrm{d}s=\int_\Omega f\,\mathrm{d}\mathbf{x}-\int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g\,\mathrm{d}s.$$

Rewriting in terms of interior edges $e \in \Gamma_{\mathcal{I}}$ we obtain

$$\sum_{e\in\Gamma_{\mathcal{I}}}\int_e\mathcal{H}(u^+,u^-,\mathbf{n})+\mathcal{H}(u^-,u^+,-\mathbf{n})\,\mathrm{d}s+\int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g\,\mathrm{d}s+\int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,u\,\mathrm{d}s=\int_\Omega f\,\mathrm{d}\mathbf{x}.$$

Hence, the discretization is **conservative**, i.e.

$$\int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g\,\mathrm{d}s+\int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,u\,\mathrm{d}s=\int_\Omega f\,\mathrm{d}\mathbf{x},$$

if and only if the numerical flux function $\mathcal{H}$ is conservative, i.e.

$$\mathcal{H}(u^+,u^-,\mathbf{n})=-\mathcal{H}(u^+,u^-,-\mathbf{n}).$$

Numerical flux functions for the linear advection equation

The **mean value flux** (or central flux):

$$\mathcal{H}_{\mathrm{mv}}(u^+, u^-, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\,\{u\}, \qquad \text{where } \{u\} = \frac{1}{2}\left(u^+ + u^-\right).$$

The **upwind flux:**

$$\mathcal{H}_{\mathrm{uw}}(u^+, u^-, \mathbf{n}) = \begin{cases} \mathbf{b}\cdot\mathbf{n}\,u^-, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x}) < 0, \text{ i.e. } \mathbf{x}\in\partial\kappa_-, \\ \mathbf{b}\cdot\mathbf{n}\,u^+, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x}) \ge 0, \text{ i.e. } \mathbf{x}\in\partial\kappa_+, \end{cases},$$

where $\partial\kappa_-$ and $\partial\kappa_+$ are the inflow and outflow boundaries of element κ:

$$\begin{aligned} \partial\kappa_- &= \{\mathbf{x}\in\partial\kappa, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}, \\ \partial\kappa_+ &= \{\mathbf{x}\in\partial\kappa, \mathbf{b}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) \ge 0\} = \partial\kappa\setminus\partial\kappa_-. \end{aligned}$$

The **generic flux:**

$$\mathcal{H}_{b_0}(u^+, u^-, \mathbf{n}) = \mathbf{b}\cdot\mathbf{n}\,\{u\} + b_0\,[u], \qquad \text{where } [u] = u^+ - u^-.$$

- represents the mean value flux for $b_0 = 0$
- represents the upwind flux for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$

The DG discretization of the linear advection equation

For $p \geq 0$ we define the space of discontinuous piecewise polynomials of degree p:

$$\begin{aligned} V_{h,p}^{d} = \{ v_h \in L^2(\Omega) : & v_h|_\kappa \circ \sigma_\kappa \in Q_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit hypercube, and} \\ & v_h|_\kappa \circ \sigma_\kappa \in P_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit simplex}, \kappa \in \mathcal{T}_h \}, \end{aligned}$$

where P_p and Q_p are the spaces of polynomials and tensor product polynomials of degree p.

Replacing $u, v \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ in the variational formulation by discrete functions $u_h, v_h \in V_h$ gives the **discrete problem**: find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h,$$

The DG discretization of the linear advection equation

For $p \geq 0$ we define the space of discontinuous piecewise polynomials of degree p:

$$\begin{aligned} V_{h,p}^d = \{v_h \in L^2(\Omega) : & v_h|_\kappa \circ \sigma_\kappa \in Q_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit hypercube, and} \\ & v_h|_\kappa \circ \sigma_\kappa \in P_p(\hat{\kappa}) \text{ if } \hat{\kappa} \text{ is the unit simplex}, \kappa \in \mathcal{T}_h\}, \end{aligned}$$

where P_p and Q_p are the spaces of polynomials and tensor product polynomials of degree p.

Replacing $u, v \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ in the variational formulation by discrete functions $u_h, v_h \in V_h$ gives the **discrete problem**: find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h,$$

Note that $V_{h,p}^d \subset H^{1,\mathbf{b}}(\mathcal{T}_h)$ and $u \in H^{1,\mathbf{b}}(\Omega) \subset H^{1,\mathbf{b}}(\mathcal{T}_h)$, but $V_{h,p}^d \not\subset H^{1,\mathbf{b}}(\Omega)$

Coercivity

Let $B_h(\cdot,\cdot)$ be given by

$$B_h(u,v) = -\int_\Omega (\mathbf{b}u)\cdot\nabla_h v\,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\mathcal{H}_{b_0}(u^+,u^-,\mathbf{n})v\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s,$$

where $\mathcal{H}_{b_0}$ represents

- the mean value flux for $b_0 = 0$
- the upwind flux for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$

Then for all $v \in H^{1,\mathbf{b}}(\mathcal{T}_h)$ we have

$$B_h(v,v) = \|c_0 v\|^2 + \sum_{e\in\Gamma_\mathcal{I}}\int_e b_0\,[v]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\,v^2\,\mathrm{d}s =: |||v|||_{b_0}^2.$$

Existence and uniqueness of a discrete solution

$$u_h(\mathbf{x}) = \sum_{0 \le j < N_h} u_j \phi_j(\mathbf{x}), \qquad \text{with basis functions} \quad \phi_j \in V_h, 0 \le j < N_h$$

Rewrite discrete problem as a linear system

$$A\mathbf{u} = \mathbf{b}, \qquad \text{with } A_{ij} = B_h(\phi_j, \phi_i), \text{ and } b_i = F(\phi_i)$$

From coercivity of $B_h(\cdot,\cdot)$ we obtain

$$\xi_i A_{ij} \xi_j = \xi_i B_h(\phi_j, \phi_i) \xi_j = B_h(\xi_j \phi_j, \xi_i \phi_i) = B_h(\boldsymbol{\xi}, \boldsymbol{\xi}) \ge \|c_0 \boldsymbol{\xi}\|^2 \ge \gamma_0 \|\boldsymbol{\xi}\|^2,$$

Thereby A is positive definite.
Hence the linear mapping $\mathbf{u} \rightarrow A\mathbf{u}$ is injective

Lemma: For a linear mapping $A : U \rightarrow W$ of finite dimensional spaces with $\dim U = \dim W$ following properties are equivalent: A is injective, A is surjective, and A is bijective.

Thereby, the linear mapping $\mathbf{u} \rightarrow A\mathbf{u}$ is bijective
and there exists a unique discrete solution $\mathbf{u} = A^{-1}\mathbf{b}$.

Stability

We have coercivity of $B_h(\cdot,\cdot)$

$$B_h(v,v) = \|c_0 v\|^2 + \sum_{e\in\Gamma_{\mathcal{I}}} \int_e b_0\,[v]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\,v^2\,\mathrm{d}s =: |||v|||_{b_0}^2.$$

and continuity of $F(\cdot)$

$$F(v) \leq C_F |||v|||_{b_0}$$

Thereby,

$$\begin{aligned} |||v|||_{b_0}^2 &= B_h(v,v) = F(v) \leq C_F |||v|||_{b_0} \\ |||v|||_{b_0} &\leq C_F \end{aligned}$$

and we have control over all terms in

$$|||v|||_{b_0}^2 = \|c_0 v\|^2 + \sum_{e\in\Gamma_{\mathcal{I}}} \int_e b_0\,[v]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\,v^2\,\mathrm{d}s \leq C_F^2,$$

with $b_0 = 0$ the mean value flux and $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$ for the upwind flux

A priori error estimate

Theorem: Let $u \in H^{p+1}(\Omega)$ be the exact solution to the linear advection equation. Furthermore, let $u_h \in V_{h,p}^d$ be the solution to

$$B_h(u_h, v_h) = F(v_h), \quad \forall v_h \in V_{h,p}^d,$$

where
$$B_h(u,v) = -\int_\Omega (\mathbf{b}u)\cdot\nabla_h v \,\mathrm{d}\mathbf{x} + \int_\Omega cuv\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}(\mathbf{b}\cdot\mathbf{n}\,\{u\} + b_0\,[u])\,v\,\mathrm{d}s + \int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,uv\,\mathrm{d}s,$$

$$F(v) = \int_\Omega fv\,\mathrm{d}\mathbf{x} - \int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,gv\,\mathrm{d}s.$$

Then, for $b_0 = \frac{1}{2}|\mathbf{b}\cdot\mathbf{n}|$, i.e. when using the *upwind flux*, we have

$$\boxed{|||u - u_h|||_{b_0} \le Ch^{p+1/2}|u|_{H^{p+1}(\Omega)},}$$

and for $b_0 = 0$, i.e. when using the *mean value flux*, we have

$$\boxed{|||u - u_h|||_{b_0} \le Ch^{p}|u|_{H^{p+1}(\Omega)},}$$

where $|||v|||_{b_0}^2 = \|c_0 v\|^2 + \sum_{e\in\Gamma_\mathcal{I}}\int_e b_0\,[v]^2\,\mathrm{d}s + \frac{1}{2}\int_\Gamma |\mathbf{b}\cdot\mathbf{n}|\,v^2\,\mathrm{d}s.$

DG discretization based on upwind

$$-\int_\Omega (\mathbf{b}u_h)\cdot\nabla_h v_h\,\mathrm{d}\mathbf{x}+\int_\Omega c u_h v_h\,\mathrm{d}\mathbf{x}+\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}\mathcal{H}(u_h^+,u_h^-,\mathbf{n})v_h^+\,\mathrm{d}s+\int_{\Gamma_+}\mathbf{b}\cdot\mathbf{n}\,u_h v_h\,\mathrm{d}s$$
$$=\int_\Omega f v_h\,\mathrm{d}\mathbf{x}-\int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g v_h\,\mathrm{d}s\quad\forall v_h\in V_{h,p}^d.$$

with the upwind flux:

$$\mathcal{H}_{\mathrm{uw}}(u^+,u^-,\mathbf{n})=\begin{cases}\mathbf{b}\cdot\mathbf{n}\,u^-, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x})<0,\ \text{i.e. } \mathbf{x}\in\partial\kappa_-,\\ \mathbf{b}\cdot\mathbf{n}\,u^+, & \text{for } (\mathbf{b}\cdot\mathbf{n})(\mathbf{x})\geq 0,\ \text{i.e. } \mathbf{x}\in\partial\kappa_+,\end{cases}$$

we obtain: find $u_h\in V_{h,p}^d$ such that

$$-\int_\Omega (\mathbf{b}u_h)\cdot\nabla_h v_h\,\mathrm{d}\mathbf{x}+\int_\Omega c u_h v_h\,\mathrm{d}\mathbf{x}+\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa_-\setminus\Gamma}\mathbf{b}\cdot\mathbf{n}\,u_h^- v_h^+\,\mathrm{d}s+\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa_+}\mathbf{b}\cdot\mathbf{n}\,u_h^+ v_h^+\,\mathrm{d}s$$
$$=\int_\Omega f v_h\,\mathrm{d}\mathbf{x}-\int_{\Gamma_-}\mathbf{b}\cdot\mathbf{n}\,g v_h\,\mathrm{d}s\quad\forall v_h\in V_{h,p}^d.$$

DG discretization based on upwind

Find $u_h \in V_{h,p}^d$ such that

$$-\int_\Omega (\mathbf{b} u_h)\cdot\nabla_h v_h \,\mathrm{d}\mathbf{x} + \int_\Omega c u_h v_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa_-\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\, u_h^- v_h^+ \,\mathrm{d}s + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa_+} \mathbf{b}\cdot\mathbf{n}\, u_h^+ v_h^+ \,\mathrm{d}s$$
$$= \int_\Omega f v_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g v_h \,\mathrm{d}s \quad \forall v_h \in V_{h,p}^d.$$

After integration by parts on each $\kappa \in \mathcal{T}_h$: Find $u_h \in V_{h,p}^d$ such that

$$\int_\Omega \left(\nabla_h\cdot(\mathbf{b} u_h) + c u_h\right) v_h \,\mathrm{d}\mathbf{x} - \boxed{\sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa_-\setminus\Gamma} \mathbf{b}\cdot\mathbf{n}\,(u_h^+ - u_h^-) v_h^+ \,\mathrm{d}s} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, u_h v_h \,\mathrm{d}s$$
$$= \int_\Omega f v_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_-} \mathbf{b}\cdot\mathbf{n}\, g v_h \,\mathrm{d}s \quad \forall v_h \in V_{h,p}^d.$$

Jump terms vanish if evaluated for **continuous** discrete functions $u_h \in V_{h,p}^c$.
This means: the **stable** DG discretization reduces to the **unstable** standard (continuous) Galerkin discretization of the linear advection equation.

DG discretization for the linear advection equation: Summary

DG/mean value flux	no control over $\sum_{e\in\Gamma_\mathcal{I}}\int_e [u_h]^2\,\mathrm{d}s$	unstable	$\mathcal{O}(h^p)$
DG/upwind flux	control over $\sum_{e\in\Gamma_\mathcal{I}}\int_e [u_h]^2\,\mathrm{d}s$	stable	$\mathcal{O}(h^{p+1/2})$

DG discretization for the linear advection equation: Summary

DG/mean value flux	no control over $\sum_{e\in\Gamma_{\mathcal{I}}}\int_e [u_h]^2\,\mathrm{d}s$	unstable	$\mathcal{O}(h^p)$
DG/upwind flux	control over $\sum_{e\in\Gamma_{\mathcal{I}}}\int_e [u_h]^2\,\mathrm{d}s$	stable	$\mathcal{O}(h^{p+1/2})$

We recall for continuous finite element methods:

standard Galerkin FEM	no control over $\|\mathbf{b}\cdot\nabla u_h\|$	unstable	$\mathcal{O}(h^p)$
streamline diffusion FEM	control over $\|\mathbf{b}\cdot\nabla u_h\|$	stable	$\mathcal{O}(h^{p+1/2})$

Outline

Poisson's equation

For $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \neq \emptyset$ consider the Dirichlet-Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g_D \in L^2(\Gamma_D)$ and $g_N \in L^2(\Gamma_N)$.
Rewrite this as a first-order system:

$$\sigma = \nabla u, \quad -\nabla \cdot \sigma = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \Gamma_D, \quad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N.$$

Multiply the first and second equation by test functions $\tau \in [H^1(\mathcal{T}_h)]^d$ and $v \in H^1(\mathcal{T}_h)$, integrate over $\kappa \in \mathcal{T}_h$ and integrate by parts

$$\int_\kappa \sigma \cdot \tau \, \mathrm{d}\mathbf{x} = -\int_\kappa u \nabla \cdot \tau \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa} u \, \tau \cdot \mathbf{n} \, \mathrm{d}s,$$
$$\int_\kappa \sigma \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\kappa f v \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa} \sigma \cdot \mathbf{n} \, v \, \mathrm{d}s.$$

The system flux formulation

Summation over all elements $\kappa \in \mathcal{T}_h$ leads to

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}\mathbf{x} = -\int_\Omega u \nabla_h \cdot \tau \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} u \, \tau \cdot \mathbf{n} \, \mathrm{d}s,$$

$$\int_\Omega \sigma \cdot \nabla_h v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \sigma \cdot \mathbf{n} \, v \, \mathrm{d}s.$$

Replace u and $\sigma = \nabla u$ on $\partial\kappa$ by **numerical flux functions** $\hat{u}$ and $\hat{\sigma}$:

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}\mathbf{x} = -\int_\Omega u \nabla_h \cdot \tau \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{u} \, \tau \cdot \mathbf{n} \, \mathrm{d}s,$$

$$\int_\Omega \sigma \cdot \nabla_h v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa} \hat{\sigma} \cdot \mathbf{n} \, v \, \mathrm{d}s.$$

The primal flux formulation

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}\mathbf{x} = -\int_\Omega u \nabla_h \cdot \tau \, \mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{u}\, \tau \cdot \mathbf{n} \, \mathrm{d}s,$$

$$\int_\Omega \sigma \cdot \nabla_h v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\sigma} \cdot \mathbf{n}\, v \, \mathrm{d}s.$$

Replace τ by $\nabla_h v$ and perform second integration by parts in the first equation:

$$\int_\Omega \sigma \cdot \nabla_h v \, \mathrm{d}\mathbf{x} = \int_\Omega \nabla_h u \cdot \nabla_h v \, \mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u} - u)\mathbf{n} \cdot \nabla_h v \, \mathrm{d}s.$$

Eliminate σ by substituting this into the second equation gives the **primal flux formulation:** find $u \in H^2(\mathcal{T}_h)$ such that

$$\hat{B}_h(u, v) = \int_\Omega f v \, \mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h),$$

where the bilinear form $\hat{B}_h(\cdot,\cdot) : H^2(\mathcal{T}_h) \times H^2(\mathcal{T}_h) \to \mathbb{R}$ is given by

$$\hat{B}_h(u, v) = \int_\Omega \nabla_h u \cdot \nabla_h v \, \mathrm{d}\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \hat{\sigma} \cdot \mathbf{n} v \, \mathrm{d}s + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} (\hat{u} - u)\mathbf{n} \cdot \nabla_h v \, \mathrm{d}s.$$

The face-based primal flux formulation

After further manipulations we obtain: find $u \in H^2(\mathcal{T}_h)$ such that

$$\hat{B}_h(u,v) = \int_\Omega fv\,\mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h), \qquad \text{where}$$

$$\hat{B}_h(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathcal{I}\cup\Gamma} \left([\![\hat{u}-u]\!] \cdot \{\!\{\nabla_h v\}\!\} - \{\!\{\hat{\sigma}\}\!\} \cdot [\![v]\!] \right) \mathrm{d}s$$
$$+ \int_{\Gamma_\mathcal{I}} \left(\{\!\{\hat{u}-u\}\!\} [\![\nabla_h v]\!] - [\![\hat{\sigma}]\!] \{\!\{v\}\!\} \right) \mathrm{d}s.$$

Definition 5.2 & 5.3: Mean value of scalar q and vector ϕ:

$$\{\!\{q\}\!\} = \tfrac{1}{2}(q^+ + q^-), \qquad \{\!\{\phi\}\!\} = \tfrac{1}{2}(\phi^+ + \phi^-), \qquad \text{on } \Gamma_\mathcal{I}$$

Jump of scalar q and vector ϕ:

$$[\![q]\!] = q^+\mathbf{n}^+ + q^-\mathbf{n}^-, \qquad [\![\phi]\!] = \phi^+ \cdot \mathbf{n}^+ + \phi^- \cdot \mathbf{n}^-, \qquad \text{on } \Gamma_\mathcal{I}$$

On the boundary Γ:

$$\{\!\{q\}\!\} = q^+, \qquad [\![q]\!] = q^+\mathbf{n}^+, \qquad \{\!\{\phi\}\!\} = \phi^+, \qquad [\![\phi]\!] = \phi^+ \cdot \mathbf{n}^+.$$

The DG discretization (fluxes still unspecified)

The **primal flux formulation:** find $u \in H^2(\mathcal{T}_h)$ such that

$$\hat{B}_h(u, v) = \int_\Omega f v \, \mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h), \qquad \text{where}$$

$$\begin{aligned}\hat{B}_h(u, v) = \int_\Omega \nabla_h u \cdot \nabla_h v \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\mathcal{I} \cup \Gamma} \left([\![\hat{u} - u]\!] \cdot \{\!\{\nabla_h v\}\!\} - \{\!\{\hat{\sigma}\}\!\} \cdot [\![v]\!] \right) \mathrm{d}s \\ + \int_{\Gamma_\mathcal{I}} \left(\{\!\{\hat{u} - u\}\!\} [\![\nabla_h v]\!] - [\![\hat{\sigma}]\!] \{\!\{v\}\!\} \right) \mathrm{d}s.\end{aligned}$$

Replace $u, v \in H^2(\mathcal{T}_h)$ by discrete functions $u_h, v_h \in V_{h,p}^d$.
The **discrete problem:** find $u_h \in V_{h,p}^d$ such that

$$\hat{B}_h(u_h, v_h) = \int_\Omega f v_h \, \mathrm{d}\mathbf{x} \quad \forall v_h \in V_{h,p}^d.$$

Consistency and adjoint consistency

Definition 5.1: The numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are said to be **consistent** if

$$\hat{u}(v) = v, \qquad \hat{\sigma}(v, \nabla v) = \nabla v, \quad \text{on } \Gamma_{\mathcal{I}} \cup \Gamma,$$

whenever v is a smooth function satisfying the Dirichlet boundary conditions.
$\hat{u}$ and $\hat{\sigma}$ are said to be **conservative** if they are **single-valued** on $\Gamma_{\mathcal{I}} \cup \Gamma$.

Consistency and adjoint consistency

Definition 5.1: The numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are said to be **consistent** if

$$\hat{u}(v) = v, \qquad \hat{\sigma}(v, \nabla v) = \nabla v, \quad \text{on } \Gamma_{\mathcal{I}} \cup \Gamma,$$

whenever v is a smooth function satisfying the Dirichlet boundary conditions.
$\hat{u}$ and $\hat{\sigma}$ are said to be **conservative** if they are **single-valued** on $\Gamma_{\mathcal{I}} \cup \Gamma$.
Theorem 5.7: The discretization is **consistent**, i.e.

$$\hat{B}_h(u, v) = \int_\Omega fv \, \mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h).$$

holds for the exact solution u to the primal problem
if and only if the numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are **consistent**.

Consistency and adjoint consistency

Definition 5.1: The numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are said to be **consistent** if

$$\hat{u}(v) = v, \qquad \hat{\sigma}(v, \nabla v) = \nabla v, \quad \text{on } \Gamma_{\mathcal{I}} \cup \Gamma,$$

whenever v is a smooth function satisfying the Dirichlet boundary conditions.
$\hat{u}$ and $\hat{\sigma}$ are said to be **conservative** if they are **single-valued** on $\Gamma_{\mathcal{I}} \cup \Gamma$.
Theorem 5.7: The discretization is **consistent**, i.e.

$$\hat{B}_h(u, v) = \int_\Omega fv \, \mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h).$$

holds for the exact solution u to the primal problem
if and only if the numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are **consistent**.
Theorem 5.12: The discretization of the **homogeneous** Dirichlet problem is **adjoint consistent**, i.e.

$$\hat{B}_h(v, z) = \int_\Omega j_\Omega v \, \mathrm{d}\mathbf{x} \quad \forall v \in H^2(\mathcal{T}_h).$$

holds for the exact solution z to the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \quad \text{on } \Gamma.$$

if and only if the numerical fluxes $\hat{u}$ and $\hat{\sigma}$ are **conservative**.

Derivation of various DG discretization methods

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathcal{I}\cup\Gamma} \left([\![\hat{u}_h - u_h]\!] \cdot \{\!\{\nabla_h v_h\}\!\} - \{\!\{\hat{\sigma}_h\}\!\} \cdot [\![v_h]\!] \right) \mathrm{d}s$$
$$+ \int_{\Gamma_\mathcal{I}} \left(\{\!\{\hat{u}_h - u_h\}\!\} [\![\nabla_h v_h]\!] - [\![\hat{\sigma}_h]\!] \{\!\{v_h\}\!\} \right) \mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x} \quad \forall v_h \in V_h,$$

where the numerical fluxes $\hat{u}_h$ and $\hat{\sigma}_h$ are given by

	on $\Gamma_\mathcal{I}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

Derivation of various DG discretization methods

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathcal{I}\cup\Gamma} \left([\![\hat{u}_h - u_h]\!] \cdot \{\!\{\nabla_h v_h\}\!\} - \{\!\{\hat{\sigma}_h\}\!\} \cdot [\![v_h]\!] \right) \mathrm{d}s$$
$$+ \int_{\Gamma_\mathcal{I}} \left(\{\!\{\hat{u}_h - u_h\}\!\} [\![\nabla_h v_h]\!] - [\![\hat{\sigma}_h]\!] \{\!\{v_h\}\!\} \right) \mathrm{d}s = \int_\Omega f v_h \,\mathrm{d}\mathbf{x} \quad \forall v_h \in V_h,$$

where the numerical fluxes $\hat{u}_h$ and $\hat{\sigma}_h$ are given by

	on $\Gamma_\mathcal{I}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{ip}}_\Gamma(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{ip}}_\Gamma(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}^{\mathrm{br2}}_\Gamma(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\sigma}(v, \nabla v) = \nabla v$ for smooth v
- Discretization of the homogeneous Dirichlet problem is adjoint consistent if $\hat{u}_h$ and $\hat{\sigma}_h$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_{\mathcal{I}}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\sigma}(v, \nabla v) = \nabla v$ for smooth v
- Discretization of the homogeneous Dirichlet problem is adjoint consistent if $\hat{u}_h$ and $\hat{\sigma}_h$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_{\mathcal{I}}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\sigma}(v, \nabla v) = \nabla v$ for smooth v
- Discretization of the homogeneous Dirichlet problem is adjoint consistent if $\hat{u}_h$ and $\hat{\sigma}_h$ single-valued

Assume that $\boldsymbol{\delta}^{\mathrm{ip}}(v) = \boldsymbol{\delta}^{\mathrm{br2}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(v) = 0$ for smooth functions v and $\delta^{\mathrm{ip}}(u_h)$, $\delta^{\mathrm{br2}}(u_h)$ single-valued

Derivation of various DG discretization methods

	on $\Gamma_{\mathcal{I}}$		on Γ_D		on Γ_N	
	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$	$\hat{u}_h$	$\hat{\sigma}_h$
BO	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\}$	$2u_h - g_D$	$\nabla_h u_h$	u_h	$g_N \mathbf{n}$
NIPG	$\{\!\{u_h\}\!\} + \mathbf{n}^+ \cdot [\![u_h]\!]$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	$2u_h - g_D$	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
SIPG	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{ip}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(u_h)$	u_h	$g_N \mathbf{n}$
BR2	$\{\!\{u_h\}\!\}$	$\{\!\{\nabla_h u_h\}\!\} - \boldsymbol{\delta}^{\mathrm{br2}}(u_h)$	g_D	$\nabla_h u_h - \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(u_h)$	u_h	$g_N \mathbf{n}$

- Discretization is consistent if $\hat{u}(v) = v$ and $\hat{\sigma}(v, \nabla v) = \nabla v$ for smooth v
- Discretization of the homogeneous Dirichlet problem is adjoint consistent if $\hat{u}_h$ and $\hat{\sigma}_h$ single-valued

Assume that $\boldsymbol{\delta}^{\mathrm{ip}}(v) = \boldsymbol{\delta}^{\mathrm{br2}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{ip}}(v) = \boldsymbol{\delta}_\Gamma^{\mathrm{br2}}(v) = 0$ for smooth functions v and $\delta^{\mathrm{ip}}(u_h)$, $\delta^{\mathrm{br2}}(u_h)$ single-valued

method of Baumann-Oden	BO	consistent	adjoint **in**consistent
non-sym. interior penalty Galerkin	NIPG	consistent	adjoint **in**consistent
symmetric interior penalty Galerkin	SIPG	consistent	adjoint consistent
2nd scheme of Bassi & Rebay	BR2	consistent	adjoint consistent

Baumann-Oden, symmetric and non-symmetric interior penalty

Choose the interior penalty term:

$$\boldsymbol{\delta}^{\mathrm{ip}}(u_h) = \delta [\![u_h]\!] \quad \text{on } \Gamma_{\mathcal{I}}, \qquad \boldsymbol{\delta}_{\Gamma}^{\mathrm{ip}}(u_h) = \delta \left(u_h - g_D \right) \mathbf{n} \quad \text{on } \Gamma_D$$

Baumann-Oden, symmetric and non-symmetric interior penalty

Choose the interior penalty term:

$$\boldsymbol{\delta}^{\mathrm{ip}}(u_h) = \delta[\![u_h]\!] \quad \text{on } \Gamma_{\mathcal{I}}, \qquad \boldsymbol{\delta}_{\Gamma}^{\mathrm{ip}}(u_h) = \delta\,(u_h - g_D)\,\mathbf{n} \quad \text{on } \Gamma_D$$

Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h, \qquad \text{where}$$

$$\begin{aligned} B_h(u,v) = & \int_\Omega \nabla_h u \cdot \nabla_h v \,\mathrm{d}\mathbf{x} \\ & + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \left(\theta[\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} - \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!]\right) \mathrm{d}s + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta[\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s, \\ F_h(v) = & \int_\Omega f v \,\mathrm{d}\mathbf{x} + \int_{\Gamma_D} \theta g_D \,\mathbf{n} \cdot \nabla v \,\mathrm{d}s + \int_{\Gamma_D} \delta g_D v \,\mathrm{d}s + \int_{\Gamma_N} g_N v \,\mathrm{d}s, \end{aligned}$$

with

method of Baumann-Oden	BO	$\theta = 1$	$\delta = 0$
non-sym. interior penalty Galerkin	NIPG	$\theta = 1$	$\delta > 0$
symmetric interior penalty Galerkin	SIPG	$\theta = -1$	$\delta > 0$

2nd method of Bassi & Rebay

Choose the penalization term:

$$\delta^{\mathrm{br2}}(u_h) = \delta_\Gamma^{\mathrm{br2}}(u_h) = -C_{\mathrm{BR2}}\{\!\{\mathbf{L}^e_{g_D}(u_h)\}\!\} \quad \text{for } e \subset \Gamma_{\mathcal{I}} \cup \Gamma_D,$$

where the so-called *local lifting operator* including Dirichlet boundary conditions, $\mathbf{L}^e_{g_D} : L^2(e) \to \underline{\Sigma}^d_{h,p}$, is a vector-valued affine operator defined by:
$\mathbf{L}^e_{g_D}(w) \in \underline{\Sigma}^d_{h,p}$ is the solution to

$$\int_\Omega \mathbf{L}^e_{g_D}(w) \cdot \tau \,\mathrm{d}\mathbf{x} = -\int_e (w - g_D)\,\mathbf{n} \cdot \tau \,\mathrm{d}s \quad \forall \tau \in \underline{\Sigma}^d_{h,p}, \qquad \text{for } e \subset \Gamma_D$$

$$\int_\Omega \mathbf{L}^e_{g_D}(w) \cdot \tau \,\mathrm{d}\mathbf{x} = -\int_e [\![w]\!] \cdot \{\!\{\tau\}\!\} \,\mathrm{d}s \quad \forall \tau \in \underline{\Sigma}^d_{h,p}, \qquad \text{on } e \subset \Gamma_{\mathcal{I}},$$

and $\mathbf{L}^e_{g_D}(w)$ is defined to be zero for $e \subset \Gamma_N$.

Here, $\underline{\Sigma}^d_{h,p} := [V^d_{h,p}]^d \subset [H^1(\mathcal{T}_h)]^d$ is a vector-valued discrete function space consisting of vector-valued polynomial functions of degree $p \geq 0$.

Continuity

- Method of Baumann-Oden (BO):

$$|B_h(u,v)| \leq |||u|||_\delta \, |||v|||_\delta, \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

for any $\delta > 0$, where

$$|||v|||_\delta^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta^{-1} \left(\mathbf{n}\cdot\{\!\{\nabla v\}\!\}\right)^2 \, ds + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta[v]^2 \, ds.$$

Continuity

- Method of Baumann-Oden (BO):

$$|B_h(u,v)| \leq |||u|||_\delta \, |||v|||_\delta, \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

  for any $\delta > 0$, where

$$|||v|||_\delta^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta^{-1} \left(\mathbf{n}\cdot\{\!\{\nabla v\}\!\}\right)^2 \, \mathrm{d}s + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta [v]^2 \, \mathrm{d}s.$$

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG):

$$|B_h(u,v)| \leq C|||u|||_\delta \, |||v|||_\delta, \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

Continuity

- Method of Baumann-Oden (BO):

$$|B_h(u,v)| \leq |||u|||_\delta \, |||v|||_\delta, \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

for any $\delta > 0$, where

$$|||v|||_\delta^2 = \|\nabla_h v\|^2_{L^2(\Omega)} + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta^{-1}\left(\mathbf{n}\cdot\{\!\{\nabla v\}\!\}\right)^2 \,\mathrm{d}s + \int_{\Gamma_\mathcal{I}\cup\Gamma_D} \delta [v]^2 \,\mathrm{d}s.$$

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG):

$$|B_h(u,v)| \leq C|||u|||_\delta \, |||v|||_\delta, \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

- 2nd method of Bassi & Rebay (BR2):

$$|B_h(u,v)| \leq C|||u|||_{L_0^e}|||v|||_{L_0^e} \qquad \forall u,v \in H^2(\mathcal{T}_h),$$

where the $|||\cdot|||_{L_0^e}$-norm is given by

$$|||v|||^2_{L_0^e} = \|\nabla_h v\|^2_{L^2(\Omega)} + \sum_{e\subset\Gamma_\mathcal{I}\cup\Gamma} \|\mathbf{L}_0^e(v)\|^2_{L^2(\Omega)}.$$

Coercivity

- Method of Baumann-Oden (BO):

$$B_h(v,v) = \|\nabla_h v\|^2_{L^2(\Omega)} \qquad \forall v \in H^2(\mathcal{T}_h),$$

But $B_h(v_h, v_h) = 0$ for $v_h \in V^d_{h,0} \subset H^2(\mathcal{T}_h)$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

Coercivity

- Method of Baumann-Oden (BO):

$$B_h(v,v) = \|\nabla_h v\|^2_{L^2(\Omega)} \qquad \forall v \in H^2(\mathcal{T}_h),$$

But $B_h(v_h, v_h) = 0$ for $v_h \in V^d_{h,0} \subset H^2(\mathcal{T}_h)$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG) with $\delta = C_{\mathrm{IP}} \frac{p^2}{h}$:

$$B_h(v_h, v_h) \geq \gamma |||v_h|||^2_\delta \qquad \forall v_h \in V^d_{h,p},$$

NIPG **stable** for $C_{\mathrm{IP}} > 0$ and SIPG **stable** for $C_{\mathrm{IP}} > C^0_{\mathrm{IP}} > 0$.

Coercivity

- Method of Baumann-Oden (BO):

$$B_h(v,v) = \|\nabla_h v\|^2_{L^2(\Omega)} \qquad \forall v \in H^2(\mathcal{T}_h),$$

  But $B_h(v_h,v_h) = 0$ for $v_h \in V^d_{h,0} \subset H^2(\mathcal{T}_h)$ and $v_h \not\equiv 0$, i.e. BO is **unstable**.

- (Non-)Symmetric interior penalty Galerkin (NIPG and SIPG) with $\delta = C_{\mathrm{IP}}\frac{p^2}{h}$:

$$B_h(v_h,v_h) \geq \gamma |||v_h|||^2_\delta \qquad \forall v_h \in V^d_{h,p},$$

  NIPG **stable** for $C_{\mathrm{IP}} > 0$ and SIPG **stable** for $C_{\mathrm{IP}} > C^0_{\mathrm{IP}} > 0$.

- 2nd method of Bassi & Rebay (BR2):

$$B_h(v,v) \geq \gamma |||v|||^2_{L^e_0} \qquad \forall v \in H^2(\mathcal{T}_h),$$

  BR2 **stable** for $C_{\mathrm{BR2}} > C^0_{\mathrm{BR2}}$ where C^0_{BR2} is the number of faces of an element ($C^0_{\mathrm{BR2}} = 3$ on triangles, $C^0_{\mathrm{BR2}} = 4$ on quadrilaterals)

A priori error estimates for NIPG and SIPG

Lemma 5.27: Let $u \in H^{p+1}(\Omega)$ be the exact solution to Poisson's equation. Furthermore, let $u_h \in V_{h,p}^d$ be the solution to

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_{h,p}^d,$$

for NIPG $(\theta = 1)$ and for SIPG$(\theta = -1)$, with $\delta = C_{\mathrm{IP}} \frac{p^2}{h}$, $C_{\mathrm{IP}} > C_{\mathrm{IP}}^0$. Then

$$\boxed{|||u - u_h|||_\delta \leq Ch^p |u|_{H^{p+1}(\Omega)}}$$

where $|||\cdot|||_\delta^2$ is the norm as defined in

$$|||v|||_\delta^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta^{-1} \left(\mathbf{n} \cdot \{\!\{\nabla v\}\!\}\right)^2 \mathrm{d}s + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} \delta [v]^2 \, \mathrm{d}s.$$

Thereby, the discretization error of the NIPG and SIPG method in the H^1-norm behaves like $\mathcal{O}(h^p)$.

Example: Model problem

Consider $\Omega = (0,1)^2$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \sin(\tfrac{1}{2}\pi x_1)\sin(\tfrac{1}{2}\pi x_2).$$

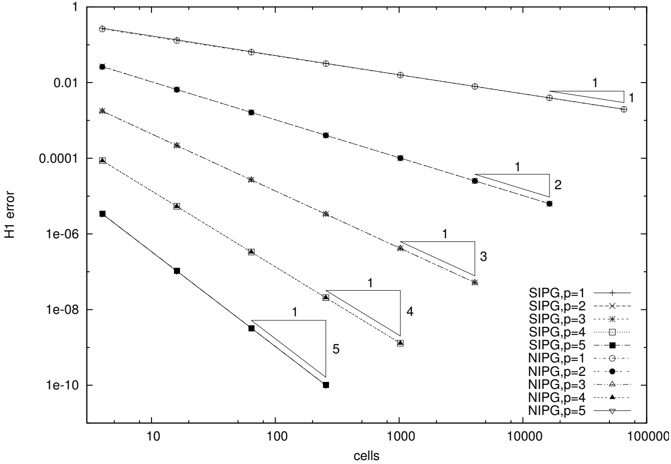
Dirichlet boundary conditions are based on the exact solution u.

Example: Model problem

Consider $\Omega = (0,1)^2$ and Poisson's equation with forcing function f such that

$$u(\mathbf{x}) = \sin(\tfrac{1}{2}\pi x_1)\sin(\tfrac{1}{2}\pi x_2).$$

Dirichlet boundary conditions are based on the exact solution u.



The H^1-error of the NIPG and SIPG methods with $p = 1, \dots, 5$, behaves like $\mathcal{O}(h^p)$

see Figure 2 on page 54

L^2-error estimate for SIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

Due to adjoint consistency of the SIPG method we have

$$B_h^s(w, z) = \int_\Omega j_\Omega w \,\mathrm{d}\mathbf{x},$$

L^2-error estimate for SIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

Due to adjoint consistency of the SIPG method we have

$$B_h^s(w,z) = \int_\Omega j_\Omega w \,\mathrm{d}\mathbf{x},$$

Choose $j_\Omega := e := u - u_h$ and assume that $z \in H^2(\Omega)$ and $\|z\|_{H^2(\Omega)} \leq C\|e\|_{L^2(\Omega)}$. Furthermore, choosing $w = e$ we obtain

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= \int_\Omega e^2 \,\mathrm{d}\mathbf{x} = B_h^s(e,z) = B_h^s(e, z - z_h) \leq |||e|||_\delta \, |||z - z_h|||_\delta, \\ &\leq Ch^p|u|_{H^{p+1}(\Omega)} Ch|z|_{H^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}\|e\|_{L^2(\Omega)}\end{aligned}$$

L^2-error estimate for SIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

Due to adjoint consistency of the SIPG method we have

$$B_h^s(w, z) = \int_\Omega j_\Omega w \, \mathrm{d}\mathbf{x},$$

Choose $j_\Omega := e := u - u_h$ and assume that $z \in H^2(\Omega)$ and $\|z\|_{H^2(\Omega)} \leq C\|e\|_{L^2(\Omega)}$. Furthermore, choosing $w = e$ we obtain

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= \int_\Omega e^2 \, \mathrm{d}\mathbf{x} = B_h^s(e, z) = B_h^s(e, z - z_h) \leq |||e|||_\delta \, |||z - z_h|||_\delta, \\ &\leq Ch^p |u|_{H^{p+1}(\Omega)} Ch|z|_{H^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}\|e\|_{L^2(\Omega)}\end{aligned}$$

Thereby,

$$\boxed{\|u - u_h\|_{L^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}}$$

i.e. the discretization error of SIPG in the L^2-norm behaves like $\mathcal{O}(h^{p+1})$.

L^2-error estimate for NIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

For NIPG (adjoint inconsistent) we have

$$B_h^n(w, z) = \int_\Omega j_\Omega w \, \mathrm{d}\mathbf{x} + 2 \int_{\Gamma_\mathcal{I} \cup \Gamma} [\![w]\!] \cdot \nabla z \, \mathrm{d}s.$$

L^2-error estimate for NIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

For NIPG (adjoint inconsistent) we have

$$B_h^n(w,z) = \int_\Omega j_\Omega w \, \mathrm{d}\mathbf{x} + 2\int_{\Gamma_\mathcal{I}\cup\Gamma} [\![w]\!] \cdot \nabla z \, \mathrm{d}s.$$

If we define $z^n \in H^2(\mathcal{T}_h)$ to be solution to

$$B_h^n(w,z^n) = \int_\Omega e w \, \mathrm{d}\mathbf{x} \qquad \forall w \in H^2(\mathcal{T}_h),$$

then z^n is mesh-dependent. In particular, $z^n \not\equiv z$ and z^n is not smooth.

L^2-error estimate for NIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad\qquad z = 0 \text{ on } \Gamma,$$

For NIPG (adjoint inconsistent) we have

$$B_h^n(w,z) = \int_\Omega j_\Omega w \,\mathrm{d}\mathbf{x} + 2\int_{\Gamma_\mathcal{I}\cup\Gamma} [\![w]\!]\cdot\nabla z\,\mathrm{d}s.$$

If we define $z^n \in H^2(\mathcal{T}_h)$ to be solution to

$$B_h^n(w,z^n) = \int_\Omega ew\,\mathrm{d}\mathbf{x} \qquad \forall w\in H^2(\mathcal{T}_h),$$

then z^n is mesh-dependent. In particular, $z^n \not\equiv z$ and z^n is not smooth.

$$\|e\|^2_{L^2(\Omega)} = \int_\Omega e^2\,\mathrm{d}\mathbf{x} = B_h^n(e,z^n) = B_h^n(e,z^n - z_h) \le |||e|||_\delta\, |||z^n - z_h|||_\delta \le Ch^p|u|_{H^{p+1}(\Omega)}$$

L^2-error estimate for NIPG

Consider the adjoint problem

$$-\Delta z = j_\Omega \quad \text{in } \Omega, \qquad z = 0 \text{ on } \Gamma,$$

For NIPG (adjoint inconsistent) we have

$$B_h^n(w,z) = \int_\Omega j_\Omega w \, \mathrm{d}\mathbf{x} + 2\int_{\Gamma_\mathcal{I}\cup\Gamma} [\![w]\!] \cdot \nabla z \, \mathrm{d}s.$$

If we define $z^n \in H^2(\mathcal{T}_h)$ to be solution to

$$B_h^n(w,z^n) = \int_\Omega e w \, \mathrm{d}\mathbf{x} \qquad \forall w \in H^2(\mathcal{T}_h),$$

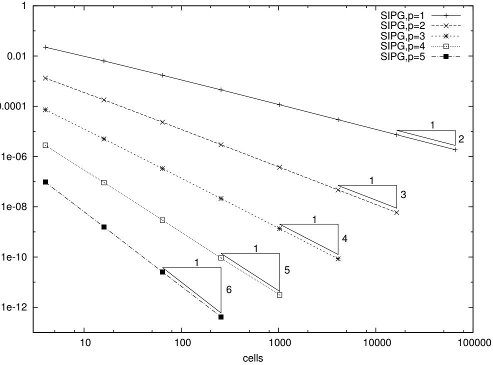
then z^n is mesh-dependent. In particular, $z^n \not\equiv z$ and z^n is not smooth.

$$\|e\|^2_{L^2(\Omega)} = \int_\Omega e^2 \, \mathrm{d}\mathbf{x} = B_h^n(e,z^n) = B_h^n(e,z^n - z_h) \le |||e|||_\delta \, |||z^n - z_h|||_\delta \le Ch^p|u|_{H^{p+1}(\Omega)}$$

Thereby, $$\boxed{\|u-u_h\|_{L^2(\Omega)} \le Ch^p|u|_{H^{p+1}(\Omega)}}$$

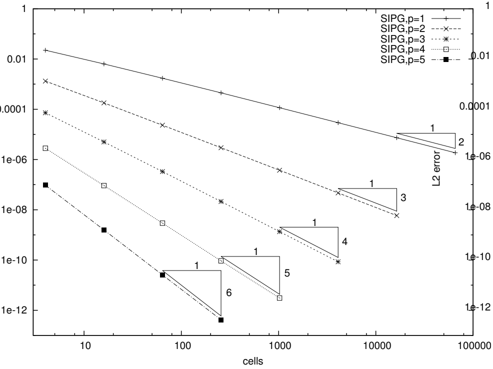
i.e. the discretization error of NIPG in the L^2-norm behaves like $\mathcal{O}(h^p)$.

Example: Model problem

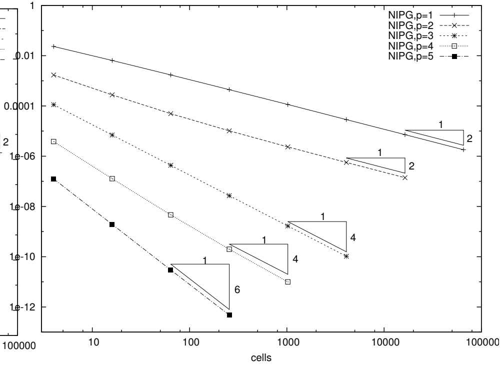


The L^2-error of the SIPG method with $p = 1, \ldots, 5$, behaves like $\mathcal{O}(h^{p+1})$

Example: Model problem

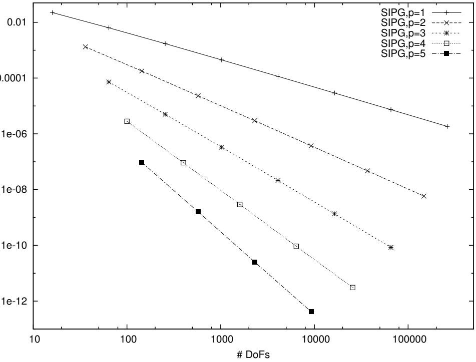


The L^2-error of the SIPG method with $p = 1, \ldots, 5$, behaves like $\mathcal{O}(h^{p+1})$



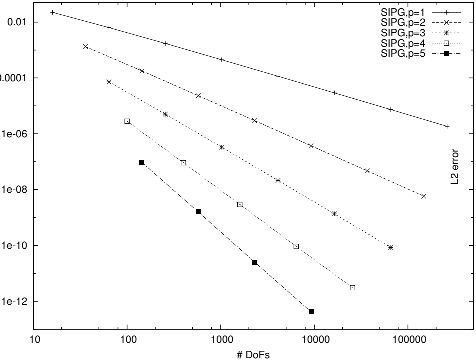
The L^2-error of the NIPG method with $p = 1, \ldots, 5$, behaves like $\mathcal{O}(h^{p+1})$ for odd p and like $\mathcal{O}(h^{p})$ for even p

Example: Model problem, computational effort

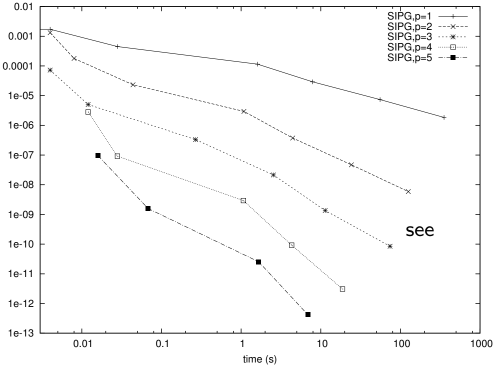


The L^2-error of the SIPG method against the number of degrees of freedom (DoFs)

Example: Model problem, computational effort



The L^2-error of the SIPG method against the number of degrees of freedom (DoFs)



see

The L^2-error of the SIPG method against the computing time in seconds

Figure 4 on page 54

DG discretizations of Poisson's equation: Summary

Consistency, adjoint consistency, stability:

Baumann-Oden	BO	consistent	adjoint **in**consistent	**un**stable
non-sym. interior penalty	NIPG	consistent	adjoint **in**consistent	stable
symmetric interior penalty	SIPG	consistent	adjoint consistent	stable
2nd Bassi & Rebay	BR2	consistent	adjoint consistent	stable

DG discretizations of Poisson's equation: Summary

Consistency, adjoint consistency, stability:

Baumann-Oden	BO	consistent	adjoint **in**consistent	**un**stable
non-sym. interior penalty	NIPG	consistent	adjoint **in**consistent	stable
symmetric interior penalty	SIPG	consistent	adjoint consistent	stable
2nd Bassi & Rebay	BR2	consistent	adjoint consistent	stable

A priori error estimates:

	H^1-error		L^2-error	
	theory	experiment	theory	experiment
NIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$ for odd p $\mathcal{O}(h^p)$ for even p
SIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$	$\mathcal{O}(h^{p+1})$

DG discretizations of Poisson's equation: Summary

Consistency, adjoint consistency, stability:

Baumann-Oden	BO	consistent	adjoint **in**consistent	**un**stable
non-sym. interior penalty	NIPG	consistent	adjoint **in**consistent	stable
symmetric interior penalty	SIPG	consistent	adjoint consistent	stable
2nd Bassi & Rebay	BR2	consistent	adjoint consistent	stable

A priori error estimates:

	H^1-error		L^2-error	
	theory	experiment	theory	experiment
NIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$ for odd p $\mathcal{O}(h^p)$ for even p
SIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$	$\mathcal{O}(h^{p+1})$

Remark: BR2 can be expected to behave like SIPG.

DG discretizations of Poisson's equation: Summary

Consistency, adjoint consistency, stability:

Baumann-Oden	BO	consistent	adjoint **in**consistent	**un**stable
non-sym. interior penalty	NIPG	consistent	adjoint **in**consistent	stable
symmetric interior penalty	SIPG	consistent	adjoint consistent	stable
2nd Bassi & Rebay	BR2	consistent	adjoint consistent	stable

A priori error estimates:

	H^1-error		L^2-error	
	theory	experiment	theory	experiment
NIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$ for odd p $\mathcal{O}(h^p)$ for even p
SIPG	$\mathcal{O}(h^p)$	$\mathcal{O}(h^p)$	$\mathcal{O}(h^{p+1})$	$\mathcal{O}(h^{p+1})$

Remark: BR2 can be expected to behave like SIPG.
L^2-error estimate of SIPG required a **duality argument** including the definition of an appropriate **adjoint problem**.